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Equiconvergence of Two Fourier Series

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The Fourier expansion of a function in the polynomials orthonormal in $[-1, 1]$ with respect to the weight function $J(x)\exp[u(x)]$, where $J(x)$ is the weight of the classical Jacobi polynomials and $u(x)$ is a real function satisfying some conditions, is studied. A comparison theorem on equiconvergence of this series with certain trigonometric Fourier series is proved. © 1995 Academic Press, Inc.

1. INTRODUCTION AND MAIN THEOREM

The generalized Jacobi polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ orthonormal in $I=[-1, 1]$ with respect to the weight function

$$Q(x)=(1-x)^{\alpha}(1+x)^{\beta}\exp[u(x)], \qquad \alpha>-1, \beta>-1 \tag{1}$$

were investigated in [1–3]. Here $J(x)=(1-x)^{\alpha}(1+x)^{\beta}$ is the weight of the classical Jacobi polynomials and $u(x)$ is a real function satisfying some conditions.

In [1, Sect. 4,2] the differential equation for the polynomial $Q_n(x)$ was derived. It has the form

$$\begin{aligned} Q^{-1}(x)[(1-x^2)\,Q_n'(x)\,Q(x)]'+(1-x^2)\,b_n(x)\,Q_n'(x) \\ +[\lambda_n^2+a_n(x)]\,Q_n(x)=0, \end{aligned} \tag{2}$$

where $\lambda_n=\sqrt{n(n+\alpha+\beta+1)}$; $a_n(x)$, $b_n(x)$ are the functions for which $|a_n(x)|<c_1 n$, $|b_n(x)|<c_2 n^{-1}$ and c_1, c_2 are constants. This equation is valid for the weight (1) with $u(x)$ satisfying the following conditions:

1. $u'''(x)$ exists in $I=[-1, 1]$.

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2. If, for brevity, we put

$$\Delta_x f(t) = \frac{f(x) - f(t)}{x - t}, \qquad v_1(t) = \Delta_x u'(t),$$

$$v_2(t) = (1 - t^2) \frac{\partial}{\partial t} \Delta_x u'(t), \qquad v_3(t) = (1 - t^2)\, \Delta_x u''(t),$$

then for $i = 1, 2, 3$, $\min(\alpha, \beta) \geqslant -\frac{1}{2}$ implies

$$\int_I (1 - t^2)^{-1/2}\, |v_i(t)|\, dt < c, \tag{3}$$

where c is a constant; cf. [1, Sect. 4,2].

The sufficient condition for (3) is proved in [1, Sect. 4,3], where it is readily seen that an arbitrary polynomial can be taken as an example for the function $u(x)$. Another example: $u(x) = \sin x$ or $u(x) = \cos x$.

Because our results are concerned with the case $\min(\alpha, \beta) \geqslant -\frac{1}{2}$, we do not mention the conditions for the case $\min(\alpha, \beta) < -\frac{1}{2}$.

In [2, Sects. 3,1, 3,2], Eq. (2) was transformed to

$$y'' + a_n(z) y = 0 \tag{4}$$

with $z = \arcsin x$, $z \in [-\pi/2, \pi/2]$, $y' = dy/dz$, $y'' = d^2y/dz^2$, $a_n(z) = \lambda_n^2 + a_n(\sin z) + \gamma(z) - \cos^2 z[b_n(\sin z) + u''(\sin z)]/2 - [b_n(\sin z) + u'(\sin z)]$ $\{[b_n(\sin z) + u'(\sin z)] \cos^2 z - 2\omega(z) \cos z - 2 \sin z\}/4$, $\omega(z) = (1 + \alpha + \beta) \operatorname{tg} z + (\alpha - \beta) \sec z$, $\gamma(z) = [\omega'(z) - \omega^2(z)/2]/2$.

The function

$$\varphi_n(z) = Q_n(\sin z) \sqrt{\cos z\, Q(\sin z)} \exp\left[1/2 \int_{\pi/2}^{z} b_n(\sin t) \cos t\, dt\right]$$

is the solution of (4); cf. [2, Sect. 3,1]. For brevity denote

$$q(z) = \sqrt{\cos z\, Q(\sin z)}, \qquad q_n(z) = Q_n(\sin z)\, q(z). \tag{5}$$

It is easily seen that $\{q_n(z)\}_{n=0}^{\infty}$ is the sequence of the functions orthonormal in $[-\pi/2, \pi/2]$ with respect to the weight $v(z) = 1$, because

$$\int_{-\pi/2}^{\pi/2} q_m(z)\, q_n(z)\, dz = \int_{-\pi/2}^{\pi/2} Q_m(\sin z)\, Q_n(\sin z) \cos z\, Q(\sin z)\, dz$$

$$= \int_{-1}^{1} Q_m(x)\, Q_n(x)\, Q(x)\, dx = \delta_{mn}, \qquad m, n = 0, 1, \ldots.$$

In the above mentioned papers properties of the coefficients $\alpha_n(z)$ and the solutions $\varphi_n(z)$ of (4) and also properties of the functions $q_n(z)$ given in (5) were analyzed. The aim of the present paper is to study, on their basis, the expansion of a function in the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$, i.e., to find conditions for the function when this expansion is equiconvergent with certain trigonometric Fourier series. The main theorem is given at the end of this section. We give its proof in Section 3 and in Section 2 we give the preliminary lemmas used in the proof.

Throughout the paper we shall use the following notation:

(i) n is a natural number or 0.

(ii) $I=[-1,1]$.

(iii) If $Q_n(x)=\sum_{k=0}^{n} a_k^{(n)} x^{n-k}$, then $p_n = a_0^{(n-1)}/a_0^{(n)}$ for $n>0$; $p_0=0$.

(iv) c_i $(i=1,2,3,...)$ are positive constants independent of n and of x (of t and of z, respectively),

(v) $\{\varepsilon_{i,n}\}_{n=1}^{\infty}$ $(i=1,2,3,...)$ is the sequence, for which $|\varepsilon_{i,n}|<c_i n^{-1/2}$.

(vi) If (a,b) is an interval then the space $L(a,b)$ is defined as usual.

Remark 1. The numbering of c_i and $\varepsilon_{i,n}$, respectively, in each lemma and theorem is independent of the numbering in the others.

Remark 2. The integrals in this paper are those of Lebesgue.

THEOREM. *Let* $\alpha>-\frac{1}{2}$, $\beta>-\frac{1}{2}$. *Let* $f(x)$ *be a function such that the integral*

$$\int_I (1-x^2)^{-1/2} f(x)\,dx = \int_{-\pi/2}^{\pi/2} f(\sin t)\,dt \tag{6}$$

exists. Let

$$s_n(x,f)=\sum_{k=0}^{n} a_k Q_k(x) \tag{7}$$

be the partial sum of the series

$$\sum_{k=0}^{\infty} a_k Q_k(x), \tag{8}$$

where $a_k=\int_I f(x)\,Q_k(x)\,Q(x)\,dx$. *Then for* $x\in(-1,1)$

$$s_n(x,f)=\int_{-\pi/2}^{\pi/2} f(\sin t)\,D_n(z,t)\,dt+\rho_n, \tag{9}$$

where $z=\arcsin x$, $\lim_{n\to\infty}\rho_n=0$, *and* $D_n(z,t)$ *is the Dirichlet kernel.*

The proof is given in Section 3.

COROLLARY. *Let* $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$. *Let* $f(x)$ *be a function such that* (6) *exists. If* $s_n(x, f)$ *is given by* (7) *and*

$$S_n(z, \psi) \tag{10}$$

is the partial sum of the trigonometric Fourier series in $[-\pi, \pi]$ *of the function*

$$\psi(z) = \begin{cases} f(\sin z) & \text{for } z \in (-\pi/2, \pi/2) \\ 0 & \text{for } z \in [-\pi, \pi] - (-\pi/2, \pi/2), \end{cases}$$

then $\lim_{n \to \infty} [s_n(x, f) - S_n(z, \psi)] = 0$. *So the series* (8) *and the trigonometric Fourier series of* $\psi(z)$ *in* $[-\pi, \pi]$ *are equiconvergent in the point* $x = \sin z$, $x \in (-1, 1)$.

Proof. Follows from the theorem.

Remark 3. In [4, Chap. IX], G. Szegö gave an equiconvergence theorem for the expansion of a function into a series of classical Jacobi polynomials and certain trigonometric series. His theorem holds for $\alpha > -1$, $\beta > -1$, $x \in (-1, 1)$ assuming for the function $f(x)$ the existence of two other integrals instead of (6). Integral (6) is the special case of both Szegö's assumptions (integrals) if $\alpha = \beta = -\frac{1}{2}$, but our theorem concerns the case $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$.

2. PRELIMINARIES

Let $s_n(x, f)$ be the partial sum of the trigonometric Fourier series in $[-\pi, \pi]$ of a function $f(x)$. Let $\delta > 0$ be a number such that $(x - \delta, x + \delta) \subset (-\pi, \pi)$ and $f(x) \in L(-\pi, \pi)$. For $S_n(x, f)$ and the Dirichlet kernel $D_n(x, t)$ we recall the well-known relations (cf. [5])

$$S_n(x, f) = \int_{x-\delta}^{x+\delta} f(t)\, D_n(x, t)\, dt + \varepsilon_n(x), \tag{11}$$

where $\lim_{n \to \infty} \varepsilon_n(x) = 0$,

$$\left| \int_{x+\delta}^{\pi} D_n(x, t)\, dt \right| < c_1 \delta^{-1} n^{-1}, \tag{12}$$

$$\left| \int_{-\pi}^{x-\delta} D_n(x, t)\, dt \right| < c_2 \delta^{-1} n^{-1} \tag{13}$$

and for $n > e$, $x \in (-\pi, \pi)$,

$$|S_n(x, f)| < k \ln n, \tag{14}$$

where k is independent of n but it depends on x.

The next lemmas describe some properties of the functions $\{q_n(z)\}_{n=0}^{\infty}$ defined in (5):

LEMMA 1. *Let* $\alpha > -1/2$, $\beta > -1/2$, $z \in [-\pi/2, \pi/2]$, $a, b \in (-\pi/2, \pi/2)$, *and* $t \in (z-\delta, z+\delta) \subset (-\pi/2, \pi/2)$ *for* $\delta > 0$. *Then*

$$|q_n(z)| < c_1, \tag{15}$$

$$\left|\int_a^b q_n(z)\, q(z)\, dz\right| < c_2 n^{-1}, \tag{16}$$

$$q_n''(t) + n^2 q_n(t) = r_n(t), \tag{17}$$

where $|r_n(t)| < c_3 n$.

Proof. Inequality (15) is proved in [1, Sect. 2,9]; (16) follows from the inequality $|\int_a^b q_n(z)\, dz| < c_4 n^{-1}$, which is proved in [3, Theorem 3,3] because $q'(z)$ is bounded and integrable on any interval $\langle a, b \rangle \subset (-\pi/2, \pi/2)$; (17) is proved in [3, Theorem 3,4].

LEMMA 2. *For every* $z \in [-\pi/2, \pi/2]$

$$\int_{-\pi/2}^{\pi/2} q_n(z, t)\, q(t)\, dt = q(z). \tag{18}$$

Proof.

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} q_n(z, t)\, q(t)\, dt &= \sum_{k=0}^{n} q_k(z) \int_{-\pi/2}^{\pi/2} q_k(t)\, q(t)\, dt \\
&= \sum_{k=0}^{n} q_k(z) \int_{-\pi/2}^{\pi/2} Q_k(\sin t)\, Q(\sin t) \cos t\, dt \\
&= \sum_{k=0}^{n} q_k(z) \int_I Q_k(x)\, Q(x)\, dx \\
&= q_0(z) \int_{-\pi/2}^{\pi/2} q_0(t)\, q(t)\, dt,
\end{aligned}$$

because

$$\int_I Q_k(x)\, Q(x)\, dx = 0 \qquad \text{for} \quad k = 1, 2, ..., n.$$

Since $Q_0(x) = a_0^{(0)}$, the integral (18) has the form

$$\int_{-\pi/2}^{\pi/2} q_n(z, t)\, q(t)\, dt = a_0^{(0)} q(z) \int_{-\pi/2}^{\pi/2} a_0^{(0)} q^2(t)\, dt = q(z)$$

as a consequence of the orthonormality of the system $\{q_n(z)\}_{n=0}^{\infty}$.

LEMMA 3. *For* $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$, $z \in (-\pi/2, \pi/2)$, *and* $j_n = (z - n^{-1/2}, z + n^{-1/2})$

$$\left| \int_{[-\pi/2, \pi/2] - j_n} q_n(z, t)\, q(t)\, dt \right| < c_1 n^{-1/2}. \tag{19}$$

Proof. Put $z = \arcsin x$, $t = \arcsin u$, $x - u = s(t)$. Using the Christoffel–Darboux formula we have

$$\begin{aligned} q_n(z, t) &= \sum_{k=0}^{n} q_k(z)\, q_k(t) \\ &= \sum_{k=0}^{n} Q_k(x)\, Q_k(u)\, q(z)\, q(t) \\ &= p_{n+1} (x - u)^{-1} [Q_{n+1}(x)\, Q_n(u) - Q_n(x)\, Q_{n+1}(u)]\, q(z)\, q(t), \end{aligned}$$

where $0 < p_{n+1} = a_0^{(n)}/a_0^{(n+1)} < 1$ (cf. [1, Sect. 2,3]). Then

$$\begin{aligned} &\int_{z+n^{-1/2}}^{\pi/2} q_n(z, t)\, q(t)\, dt \\ &\quad = p_{n+1} \int_{z+n^{-1/2}}^{\pi/2} [q_{n+1}(z)\, q_n(t) - q_n(z)\, q_{n+1}(t)]\, s^{-1}(t)\, q(t)\, dt \\ &\quad = p_{n+1} s^{-1}(z + n^{-1/2}) \int_{z+n^{-1/2}}^{\zeta} [q_{n+1}(z)\, q_n(t) - q_n(z)\, q_{n+1}(t)]\, q(t)\, dt, \end{aligned}$$

where the second mean-value theorem of the integral calculus was used. Here $\zeta \in (z + n^{-1/2}, \pi/2)$ and the function $-s^{-1}(t)$ is obviously decreasing in the interval $(z, \pi/2)$. To estimate our integral we use (15) and (16) and we obtain

$$\begin{aligned} \left| \int_{z+n^{-1/2}}^{\pi/2} q_n(z, t)\, q(t)\, dt \right| &< |s^{-1}(z + n^{-1/2})| (c_2 n^{-1} + c_3 n^{-1}) \\ &= |s^{-1}(z + n^{-1/2})|\, c_4 n^{-1}. \end{aligned} \tag{20}$$

Further

$$\begin{aligned}|s^{-1}(z+n^{-1/2})| &= \left|\frac{1}{\sin(z+n^{-1/2})-\sin z}\right| \\ &= \left|\frac{1}{2\sin(\frac{1}{2}n^{-1/2})\cos(z+\frac{1}{2}n^{-1/2})}\right| \\ &< \left|\frac{n^{1/2}}{2\cos(z+\frac{1}{2}n^{-1/2})}\right| \\ &< c_5 n^{1/2}. \end{aligned} \tag{21}$$

From (20) and (21) we get

$$\left|\int_{z+n^{-1/2}}^{\pi/2} q_n(z,t)\, q(t)\, dt\right| < c_4 n^{-1} c_5 n^{1/2} = c_6 n^{-1/2}. \tag{22}$$

Similarly we prove

$$\left|\int_{-\pi/2}^{z-n^{-1/2}} q_n(z,t)\, q(t)\, dt\right| < c_7 n^{-1/2}. \tag{23}$$

The inequality (19) follows from (22) and (23).

LEMMA 4. *Let $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$. Assume that $f(x) \in L(-\pi/2, \pi/2)$, $z \in (-\pi/2, \pi/2)$, and $\delta > 0$ is a number such that $-\pi/2 < z-\delta < z+\delta < \pi/2$. Then*

$$\lim_{n\to\infty} \int_{z+\delta}^{\pi/2} f(t)\, q_n(z,t)\, q(t)\, dt = 0$$

and

$$\lim_{n\to\infty} \int_{-\pi/2}^{z-\delta} f(t)\, q_n(z,t)\, q(t)\, dt = 0.$$

Proof. We use the Christoffel–Darboux formula and the Riemann–Lebesgue theorem, which holds also for uniformly bounded orthogonal systems (cf. [5, Chap. II, Theorems (4.4), (4.6)]). The inequality (15) shows that the system $\{q_n(t)\}_{n=0}^{\infty}$ is uniformly bounded in $[-\pi/2, \pi/2]$.

LEMMA 5. *Let $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$. Let $z \in (-\pi/2, \pi/2)$ and $\delta > 0$ be a number such that $j_\delta = (z-\delta, z+\delta) \subset (-\pi/2, \pi/2)$. Then for every $t \in j_\delta$*

$$q_n(z,t)\, q(t) = [q(z) + \varepsilon_{1,n}]\, D_n(z,t) + \sigma_n(t), \tag{24}$$

where $\{\sigma_n(t)\}_{n=0}^{\infty}$ is a uniformly bounded sequence in j_δ and $D_n(z, t)$ is the Dirichlet kernel.

Proof. We use the differential equation (17) and the Christoffel–Darboux formula, which imply

$$s(t)\, q_n(z, t) = p_{n+1}[q_{n+1}(z)\, q_n(t) - q_n(z)\, q_{n+1}(t)],$$

where $s(t) = \sin z - \sin t$, $0 < p_{n+1} < 1$. Denote $\psi_n(t) = s(t)\, q_n(z, t)$. Then we have

$$\psi_n''(t) = p_{n+1}[q_{n+1}(z)\, q_n(t) - q_n(z)\, q_{n+1}(t)]. \tag{25}$$

According to (17)

$$q_n''(t) = -(n+\tfrac{1}{2})^2\, q_n(t) + \beta_n(t), \tag{26}$$

$$q_{n+1}''(t) = -(n+\tfrac{1}{2})^2\, q_{n+1}(t) + \gamma_n(t), \tag{27}$$

where

$$\beta_n(t) = r_n(t) + (n+\tfrac{1}{4})\, q_n(t), \qquad \gamma_n(t) = r_{n+1}(t) + (n+\tfrac{1}{4})\, q_{n+1}(t)$$

with estimates

$$|\beta_n(t)| < c_1 n + c_2 n = c_3 n \tag{28}$$

and

$$|\gamma_n(t)| < c_1(n+1) + c_2 n < c_4 n. \tag{29}$$

Substituting (26) and (27) into (25) we get the differential equation

$$\psi_n''(t) + (n+\tfrac{1}{2})^2\, \psi_n(t) = \delta_n(t), \tag{30}$$

where $\delta_n(t) = p_{n+1}[q_{n+1}(z)\, \beta_n(t) - q_n(z)\, \gamma_n(t)]$. The solution of (30) is the function

$$\psi_n(t) = v_n \sin[(n+\tfrac{1}{2})(z-t)] + \eta_n(t), \tag{31}$$

where v_n for $n = 1, 2, 3, \ldots$ are constants and

$$\eta_n(t) = \frac{2}{2n+1} \int_z^t \delta_n(u) \sin[(n+\tfrac{1}{2})(t-u)]\, du.$$

Now we need to estimate v_n and the function $s^{-1}(t)\, \eta_n(t)$.

The estimate of v_n: from (31)

$$\begin{aligned} q_n(z, z) &= \lim_{t \to z} s^{-1}(t)\, \psi_n(t) \\ &= v_n \lim_{t \to z} \frac{\sin[(n+\frac{1}{2})(z-t)]}{\sin z - \sin t} + \lim_{t \to z} \frac{\eta_n(t)}{\sin z - \sin t} \\ &= v_n \lim_{t \to z} \frac{(n+\frac{1}{2}) \cos[(n+\frac{1}{2})(z-t)]}{\cos t} + \frac{\eta_n'(z)}{-\cos z} \\ &= v_n \frac{n+\frac{1}{2}}{\cos z}, \end{aligned}$$

hence

$$0 < v_n = \frac{q_n(z, z) \cos z}{n+\frac{1}{2}} = \frac{\cos z \sum_{k=0}^{n} q_k^2(z)}{n+\frac{1}{2}} < c_5.$$

The estimate of $s^{-1}(t)\, \eta_n(t)$: from (28), (29), and (15) for $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$

$$|\delta_n(t)| < p_{n+1}[|\beta_n(t)|\, |q_{n+1}(z)| + |\gamma_n(t)|\, |q_n(z)|] < c_6 n,$$

hence $|\eta_n(t)| < c_7\, |z-t|$. From this

$$|s^{-1}(t)\, \eta_n(t)| < c_7 \frac{|z-t|}{|\sin z - \sin t|} = c_7 \frac{|z-t|}{|z-t| \cos \xi} = \frac{c_7}{\cos \xi},$$

where the Lagrange mean-value theorem was used and ξ is a number between z and t, i.e., $\xi \in j_\delta$. Hence we get

$$|s^{-1}(t)\, \eta_n(t)| < c_8. \tag{32}$$

Further, we establish the constants v_n: Equation (31) yields

$$q_n(z, t)\, q(t) = \pi v_n q(z)\, D_n(z, t) \sec z + \chi_n(t) + s^{-1}(t)\, \eta_n(t)\, q(t),$$

where

$$\chi_n(t) = v_n\{s^{-1}(t) \sin[(n+\tfrac{1}{2})(z-t)]\, q(t) - \pi q(z)\, D_n(z, t) \sec z\},$$

where for $t \in j_\delta$ and for almost all natural numbers

$$\begin{aligned} |\chi_n(t)| &= |v_n| \left| \frac{\sin[(n+\frac{1}{2})(z-t)]}{2 \sin((z-t)/2)} \left[\frac{q(t)}{\cos((z+t)/2)} - \frac{q(z)}{\cos z} \right] \right| \\ &< c_8 \left| q(t) \sec \frac{z+t}{2} - q(z) \sec z \right| \left| \operatorname{cosec} \frac{z-t}{2} \right| < c_9, \end{aligned} \tag{33}$$

because the function $g(t) = q(t) \sec((z+t)/2) - q(z) \sec z$ has the finite derivative

$$g'(z) = \frac{1}{2} \lim_{t \to z} \frac{g(t)}{\sin((t-z)/2)}.$$

Then

$$q_n(z, t)\, q(t) = \pi v_n q(z)\, D_n(z, t) \sec z + \sigma_n(t), \tag{34}$$

where according to (32) and (33) for $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$

$$|\sigma_n(t)| = |\chi_n(t) + s^{-1}(t)\, \eta_n(t)\, q(t)| < c_{10}. \tag{35}$$

Using the relations (18), (19), (12), and (13), where we put $\delta = n^{-1/2}$, and the relation (35) from the equality

$$\int_{j_n} q_n(z, t)\, q(t)\, dt = \pi v_n q(z) \sec z \int_{j_n} D_n(z, t)\, dt + \int_{j_n} \sigma_n(t)\, dt$$

we get the relation

$$q(z) + \varepsilon_{2,n} = \pi v_n q(z) \sec z (1 + \varepsilon_{3,n}) + \varepsilon_{4,n}, \tag{36}$$

because

$$\left| \int_{j_n} \sigma_n(t)\, dt \right| < [z + n^{-1/2} - (z - n^{-1/2})]\, c_{10} = 2c_{10} n^{-1/2}.$$

From (36) we obtain $\pi v_n q(z) \sec z = q(z) + \varepsilon_{2,n} - \varepsilon_{4,n} - \varepsilon_{5,n}$, i.e.,

$$v_n = \frac{\cos z}{\pi} + \varepsilon_{6,n},$$

and substituting it into (34) we get (24).

3. Proof of Theorem

Assume that $f(x)$ is a function for which (6) exists. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for the interval $j_\delta = (z - \delta, z + \delta) \subset (-\pi/2, \pi/2)$

$$\int_{j_\delta} |f(\sin t)|\, dt < \varepsilon. \tag{37}$$

Then

$$s_n(x, f) = \int_I f(u)\, Q_n(x, u)\, Q(u)\, du = q^{-1}(z) \int_{-\pi/2}^{\pi/2} f(\sin t)\, q_n(z, t)\, q(t)\, dt,$$

where $z = \arcsin x$, $t = \arcsin u$.

Using the relation (24) we get

$$\begin{aligned} s_n(x, f) = (1 + \varepsilon_{1,n}) &\int_{j_\delta} f(\sin t)\, D_n(z, t)\, dt \\ &+ q^{-1}(z) \int_{[-\pi/2, \pi/2] - j_\delta} f(\sin t)\, q_n(z, t)\, q(t)\, dt \\ &+ \int_{j_\delta} f(\sin t)\, q^{-1}(z)\, \sigma_n(t)\, dt. \end{aligned} \tag{38}$$

Since according to (5) and (35), $|q^{-1}(z)\, \sigma_n(t)| < c_1$ in j_δ, taking into account (37), the following inequality is valid:

$$\left| \int_{j_\delta} f(\sin t)\, q^{-1}(z)\, \sigma_n(t)\, dt \right| < c_1. \tag{39}$$

Construct the function $\psi(z)$ as follows:

$$\psi(z) = \begin{cases} f(\sin z) & \text{for } z \in (-\pi/2, \pi/2) \\ 0 & \text{for } z \in [-\pi, \pi] - (-\pi/2, \pi/2). \end{cases}$$

If $S_n(z, \psi)$ is given by (10), then according to (11) we express it in the form

$$\begin{aligned} S_n(z, \psi) &= \int_{-\pi/2}^{\pi/2} f(\sin t)\, D_n(z, t)\, dt \\ &= \int_{j_\delta} f(\sin t)\, D_n(z, t)\, dt + \varepsilon_n(z), \end{aligned} \tag{40}$$

where

$$\lim_{n \to \infty} \varepsilon_n(z) = 0. \tag{41}$$

Subtracting (40) from (38) for $x = \sin z$ we have

$$\begin{aligned}\rho_n &= s_n(x, f) - S_n(z, \psi)\\ &= \varepsilon_{1,n} \int_{j_\delta} f(\sin t)\, D_n(z, t)\, dt\\ &\quad + q^{-1}(z) \int_{[-\pi/2, \pi/2] - j_\delta} f(\sin t)\, q_n(z, t)\, q(t)\, dt\\ &\quad + \int_{j_\delta} f(\sin t)\, q^{-1}(z)\, \sigma_n(t)\, dt - \varepsilon_n(z),\end{aligned}$$

where

$$\begin{aligned}\left|\varepsilon_{1,n} \int_{j_\delta} f(\sin t)\, D_n(z, t)\, dt\right| &\leqslant |\varepsilon_{1,n}|\, [S_n(z, \psi) + |\varepsilon_n(z)|]\\ &< c_2 n^{-1/2} [c_3 \ln n + |\varepsilon_n(z)|]\end{aligned}$$

with regard to (40) and (14), where

$$\lim_{n \to \infty} \left|\varepsilon_{1,n} \int_{j_\delta} f(\sin t)\, D_n(z, t)\, dt\right| = 0. \tag{42}$$

Further

$$\begin{aligned}\lim_{n \to \infty} \sup |\rho_n| &\leqslant \lim_{n \to \infty} \left|\varepsilon_{1,n} \int_{j_\delta} f(\sin t)\, D_n(z, t)\, dt\right|\\ &\quad + \lim_{n \to \infty} \left|q^{-1}(z) \int_{[-\pi/2, \pi/2] - j_\delta} f(\sin t)\, q_n(z, t)\, q(t)\, dt\right|\\ &\quad + \lim_{n \to \infty} |\varepsilon_n(z)|\\ &\quad + \lim_{n \to \infty} \sup \left|\int_{j_\delta} f(\sin t)\, q^{-1}(z)\, \sigma_n(t)\, dt\right|.\end{aligned}$$

The first three limits on the right-hand side of this inequality are 0 in consequence of (42), Lemma 4, and (41). Hence, with regard to (39), $\lim_{n \to \infty} \sup |\rho_n| \leqslant c_1$ and as ε is any positive number, it yields

$$\lim_{n \to \infty} \sup |\rho_n| = 0,$$

i.e.,

$$\lim_{n \to \infty} \rho_n = 0.$$

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